A note on projections of Gibbs measures from a class arising in economic modeling

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Abstract: A result about projections of Gibbs measures from a particular class arising in economic modeling is proved.

1 Introduction

Though individual human behavior undoubtedly is far more complex than the behavior of individual objects in Physics, in the study of coordination phenomena in large economic systems a reasonable approach is to restrict attention to certain known individual behavioral regularities of consumers, traders etc which are simple and stable enough to allow Statistical-Mechanics-based modeling. Such an approach leaves the state of any single individual random, and makes instead the *collective* behavior of agents endogenous in a model, given the a-priori known individual behavioral regularities which are formalized as conditional probability distributions of individual variables [1]. Thus it corresponds to the Dobrushin-Lanford-Ruelle approach to defining Gibbs measures on countably-infinite structures [2].

A natural property of models in Economics is that typically multiple individual variables of distinct types must be introduced to characterize a *single* individual economic entity ("agent"), and the macroscopic variables of the system are determined by the interaction of all variables of all agents. Taking the statistical approach, one is thus led to Gibbsian fields with multiple types of variables, with interactions both between variables of the same type associated to different agents and variables of different types associated to the same agent. Since not all types of variables are of immediate economic relevance, one is typically interested in projected infinite-volume Gibbs measures corresponding to a given subset of variable types. In that context, the present note provides a result which for a specific structure of interactions simplifies the computation of the projected measure.

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2 The result

Let $\mathbb{Z}^d$ denote the set of agents in a large economy. To each agent $i \in \mathbb{Z}^d$ there are associated two variables x_i and y_i with values in X and Y, respectively. For concreteness, we consider the case $X = Y = \mathbb{R}^n$. We set $\Omega_X = \Omega_Y := (\mathbb{R}^n)^{\mathbb{Z}^d}$ and $\Omega := \Omega_X \times \Omega_Y = (\mathbb{R}^n \times \mathbb{R}^n)^{\mathbb{Z}^d}$. By $\mathcal{B}(\Omega_X) = \mathcal{B}(\Omega_Y)$ and $\mathcal{B}(\Omega)$ we denote the corresponding Borel σ-algebras of these spaces. For any $\Lambda \subset \mathbb{Z}^d$ we define $\Omega_X^\Lambda = \Omega_Y^\Lambda := (\mathbb{R}^n)^\Lambda$ and $\Omega_\Lambda := \Omega_X^\Lambda \times \Omega_Y^\Lambda$. An element of Ω will be denoted by $x \times y$.

A Gibbs measure μ on Ω appropriately represents an equilibrium state of a large multi-component system with some given structure of local interactions between them. For any finite $\Lambda \subset \mathbb{Z}^d$ and any $\bar{x} \times \bar{y} \in \Omega$ the corresponding conditional Gibbs measures in finite volumes are of the form

$$\mu_\Lambda \left(dx_\Lambda \times dy_\Lambda \,|\, \bar{x} \times \bar{y}\right) = \frac{1}{Z_\Lambda(\bar{x} \times \bar{y})} p_\Lambda \left(x_\Lambda \times y_\Lambda \,|\, \bar{x} \times \bar{y}\right) dx_\Lambda dy_\Lambda,$$

where

$$Z_\Lambda(\bar{x} \times \bar{y}) := \int_{\Omega_\Lambda} p_\Lambda \left(x_\Lambda \times y_\Lambda \,|\, \bar{x} \times \bar{y}\right) dx_\Lambda dy_\Lambda$$

is the so-called partition function, and $p_\Lambda \left(x_\Lambda \times y_\Lambda \,|\, \bar{x} \times \bar{y}\right)$ the conditional density for variables in Λ derived from the Λ-Hamiltonian given the configuration $(\bar{x} \times \bar{y})_{\Lambda^c}$.

Motivated by certain economic models, we consider Gibbs measures from the following class:

Definition 2.1 *Let $\mathcal{G}_0$ denote the class of Gibbs measures on Ω, whose corresponding conditional distributions do not depend on condition from Ω_X, i.e.*

$$p_\Lambda \left(x_\Lambda \times y_\Lambda \,|\, \bar{x} \times \bar{y}\right) = p_\Lambda \left(x_\Lambda \times y_\Lambda \,|\, \bar{y}\right)$$

Example 2.1 *The following conditional densities fulfill the condition in the above definition (see [3] for the economic motivation behind this particular interaction structure).*

$$p_\Lambda \left(x_\Lambda \times y_\Lambda \,|\, \bar{x} \times \bar{y}\right) =$$

$$= \exp\left\{ -\sum_{i\in\Lambda} x_i^2 - J_c \sum_{i\in\Lambda} (x_i - y_i)^2 - J_s \sum_{\langle i,\, j\rangle \in\Lambda} (y_i - y_j)^2 - J_s \sum_{\langle i,\, j\rangle \in \mathbb{Z}^d,\ i\in\Lambda,\ j\in\Lambda^c} (y_i - \bar{y}_j)^2 \right\},$$

with a finite $\Lambda \subset \mathbb{Z}^d$ and $\langle i,\, j\rangle$ denoting all $i,\, j$ such that $|i-j| = 1$.

Since only a subset of variable-types is of direct economic relevance, one typically is led to the problem of computing certain projected infinite-volume Gibbs measures. In the specific context specified above, we obtain the following result about the projected measure.

Theorem 2.1 *Suppose that $\mu \in \mathcal{G}_0$. Then the measure μ^Y_{eff} which is defined on Ω_Y by*

$$\mu^Y_{eff}(A) = \mu(\Omega_X \times A), \quad A \in \mathcal{B}(\Omega_Y)$$

will be a Gibbs measure whose corresponding conditional measures in finite volume $\Lambda \subset \mathbb{Z}^d$ are given by

$$\mu_{\Lambda,\,eff}(dy_\Lambda \,|\, \bar{y}) = \int_{\Omega_X^\Lambda} \mu_\Lambda(dx_\Lambda \times dy_\Lambda \,|\, \bar{y}) = \tag{1}$$

$$= \int_{\Omega_X^\Lambda} \mu_\Lambda(dx_\Lambda \times dy_\Lambda \,|\, \bar{x} \times \bar{y}), \quad \bar{x} \times \bar{y} \in \Omega.$$

PROOF: Let us consider Gibbs specifications which corresponds to the measures μ^Y_{eff}, for any $\Lambda \subset \mathbb{Z}^d$-finite and $\bar{y} \in \Omega$ given by the probability kernel

$$\pi^Y_\Lambda(A \,|\, \bar{y}) = \int_{A'} \mu_{\Lambda,\,eff}(dy_\Lambda \,|\, \bar{y}), \tag{2}$$

where $A' := \left\{ y \in \Omega^\Lambda_Y \,|\, y \times \bar{y}_{\Lambda^c} \in A \right\}$, and $A \in \mathcal{B}(\Omega_Y)$. Equation 1 implies now the following

$$\pi^Y_\Lambda(A \,|\, \bar{y}) = \int_{A'} \int_{\Omega_X^\Lambda} \mu_\Lambda(dx_\Lambda \times dy_\Lambda \,|\, \bar{y}) =$$

$$= \pi_\Lambda(\Omega_X \times A \,|\, \bar{y}),$$

where $\pi_\Lambda(\cdot \,|\, \bar{y})$ are the Gibbs specifications corresponding to the measure μ. We show that the DLR-equations for the measure μ^Y_{eff} hold.

$$(\mu^Y_{eff}\pi^Y_\Lambda)(A) = \int_{\Omega_Y} \pi^Y_\Lambda(A|\, \bar{y})\mu^Y_{eff}(d\bar{y}) = \int_{\Omega_Y} \pi_\Lambda(\Omega_X \times A|\, \bar{y})\mu(\Omega_X \times d\bar{y}) =$$

$$= \int_{\Omega_X} \int_{\Omega_Y} \pi_\Lambda(\Omega_X \times A|\, \bar{y})\mu(d\bar{x} \times d\bar{y}).$$

But by the DLR-equations for the measure μ the latter expression is indeed equal to

$$\mu(\Omega_X \times A) = \mu^Y_{eff}(A).$$

References

[1] Föllmer H., Random Economies with Many Interacting Agents, *Journal of Mathematical Economics* 1, 51-62 (1974).

[2] Georgii H.-O., *Gibbs measures and Phase Transitions*, De Gruyter (1988).

[3] Hohnisch M., *From Large Economies with Microscopic Uncertainty to the Concept of Statistical Modeling*, University of Bonn, work in progress (2006).